

THE SHARP AFFINE L^2 SOBOLEV TRACE INEQUALITY AND VARIANTS

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ABSTRACT. We establish a sharp affine L^p Sobolev trace inequality by using the L_p Busemann-Petty centroid inequality. For $p = 2$, our affine version is stronger than the famous sharp L^2 Sobolev trace inequality proved independently by Escobar and Beckner. Our approach allows also to characterize all cases of equality in this case. For this new inequality, no Euclidean geometric structure is needed.

1. INTRODUCTION AND STATEMENTS

Sobolev inequalities are among the fundamental tools connecting analysis and geometry. A specially important inequality, the sharp L^2 Sobolev trace inequality on the half-space $\mathbb{R}_+^n := \mathbb{R}_+ \times \mathbb{R}^{n-1}$, $n \geq 3$, states that

$$(1) \quad \left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{2(n-1)}{n-2}} dx \right)^{\frac{n-2}{n-1}} \leq \mathcal{K}_n \int_{\mathbb{R}_+^n} |\nabla f(t, x)|^2 dx dt$$

for all measurable functions $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ with L^2 -integrable gradient on \mathbb{R}_+^n , where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^n in the right-hand side.

The existence of extremal functions for (1) was proved by Lions in [25] by using concentration–compactness principle. The classification of them was established, independently, by Escobar in [11] and Beckner in [2] by using the conformal invariance satisfied by this inequality. The central tool used by Escobar is the Obata’s Rigidity Theorem, while Beckner reformulated the inequality dually in terms of fractional Sobolev spaces and used the sharp Hardy-Littlewood-Sobolev inequality. Precisely, equality in (1) holds if, and only if, $f(t, x)$ has the form

$$f(t, x) = \gamma \left((t + \delta)^2 + |x - x_0|^2 \right)^{-\frac{n-2}{2}}$$

for some real constants γ and δ with $\delta > 0$ and a point $x_0 \in \mathbb{R}^{n-1}$. In particular, the value of the best constant \mathcal{K}_n is given by

$$\mathcal{K}_n = \frac{1}{\sqrt{\pi}(n-2)} \left(\frac{\Gamma(n)}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \right)^{\frac{1}{n-1}}.$$

The L^p version of the sharp Sobolev trace inequality states that

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$$(2) \quad \left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{p(n-1)}{n-p}} dx \right)^{\frac{n-p}{n-1}} \leq \mathcal{K}_{n,p} \int_{\mathbb{R}_+^n} |\nabla f(t, x)|^p dx dt$$

for all measurable functions $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ with L^p -integrable gradient on \mathbb{R}_+^n . In [24], by using a mass transport method, Nazaret exhibited extremal functions to this inequality using an arbitrary norm of the gradient for any $1 < p < n$.

The literature related to (1) is extremely vast. Actually, its initial motivation came from differential geometry with the study of the Yamabe problem on compact manifolds with boundary by Escobar in [12, 13], paving the way for a series of related works, see for example [3, 21, 22], among other important references. More recently, the inequality (1) has played a relevant role in analysis, as for instance in the treatment of non-local elliptic boundary problems involving the square root of the Laplace operator on Euclidean domains. We list for example [4, 5, 6, 7, 32] among the first references in a rich and growing literature. Furthermore, we mention that the knowledge of minimizers and best constants for the trace inequalities is useful, for instance, in order to obtain existence results for non linear Steklov-like eigenvalue problems with critical growth in domains, see [15]. Among other recent works on a slightly different setting we also mention for example [9, 10].

An important inequality related to the sharp L^p Sobolev trace inequality is the sharp L^p Sobolev inequality established by Federer and Fleming [14], Fleming and Rishel [16] and Maz'ja [23] for $p = 1$ and, independently, by Aubin [1] and Talenti [33] for $1 < p < n$. A stronger version of that latter, the sharp *affine* L^p Sobolev inequality, was introduced and proved by Zhang [34] for $p = 1$ and by Lutwak, Yang and Zhang [28] for $1 < p < n$. We refer to [17], [18] and [19] for other recently obtained affine versions of classical inequalities.

In the present work, we prove a sharp affine L^2 Sobolev trace inequality on \mathbb{R}_+^n . The new trace inequality is significantly stronger than and implies the sharp L^2 Sobolev trace inequality of Escobar and Beckner. A remarkable feature of the new inequality is its independence of the norm chosen for the Euclidean space. In other words, such an inequality depends only on the vector space structure and Lebesgue measure of \mathbb{R}^n , so that it is invariant under all affine transformations of \mathbb{R}_+^n . It is surprising the existence of this new inequality because the sharp L^2 Sobolev trace inequality relies strongly on the Euclidean geometric structure of \mathbb{R}^n .

Isoperimetric inequalities are quite useful tools to compare parameters associated generally to convex bodies. In recent years a lot of attention has been paid to affine counterparts of the classical isoperimetric inequality on \mathbb{R}^n that remain invariant under the action of affine transformations. Among these ones, it is known that the Petty projection inequality for C^1 domains is equivalent to the sharp affine L^1 Sobolev inequality, see [34]. For $1 < p < n$, the sharp affine L^p Sobolev inequality was obtained by using essentially the L^p extension of the Minkowski problem, Pólya-Szegő type results and the L^p Petty projection inequality, see [28]. The strong connection between the Petty projection and the Busemann-Petty (see [26]) inequalities makes the use of the latter in this type of approach far from surprising, however, it provides us with some additional geometric information.

For the sharp affine L^2 Sobolev trace inequality, new ingredients are needed. The isoperimetric inequality needed is the L^p Busemann-Petty centroid inequality firstly proved by Lutwak, Yang and Zhang [27]. This inequality generalizes the classical Busemann-Petty centroid inequality due to Petty [30] that compares the ratio between the volume of a convex body K and that of its centroid body. A more recent proof was given

in [8] using the technique of shadow systems. The approach used here has been recently introduced in [20] and consists in connecting Sobolev type inequalities for general norms to affine Sobolev type inequalities via the L^p Busemann-Petty centroid inequality. Our method reveals in an explicit and elementary way the geometric nature behind affine inequalities.

In order to state our main theorem, a little bit of notation should be introduced.

Let $n \geq 3$. We shall denote each point of the whole space \mathbb{R}^n or half-space $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ by (t, x) or y and each point of the Euclidean sphere \mathbb{S}^{n-2} in \mathbb{R}^{n-1} by ξ .

For smooth functions $f(t, x)$ with compact support on \mathbb{R}^n we denote by $\frac{\partial f}{\partial t}$ and $\tilde{\nabla} f$ respectively the partial derivative with respect to the variable t and the gradient with respect to the variable x . Also $\nabla_\xi f$ stands for the directional derivative of f with respect to the second variable in the direction of $\xi \in \mathbb{S}^{n-2}$, namely $\nabla_\xi f = \tilde{\nabla} f \cdot \xi$.

We introduce the new integral term

$$\begin{aligned} \mathcal{E}(f) &:= c_{n-1} \left(\int_{\mathbb{S}^{n-2}} \|\nabla_\xi f\|_{L^2(\mathbb{R}_+^n)}^{1-n} d\xi \right)^{\frac{1}{1-n}} \\ &= c_{n-1} \left(\int_{\mathbb{S}^{n-2}} \left(\int_{\mathbb{R}_+^n} |\nabla_\xi f(y)|^2 dy \right)^{-\frac{n-1}{2}} d\xi \right)^{-\frac{1}{n-1}}, \end{aligned}$$

where

$$c_n = n^{\frac{n+2}{2n}} \sqrt[n]{\omega_n}.$$

Here ω_n denotes the volume of the unit Euclidean ball \mathbb{B}^n in \mathbb{R}^n . It is known that

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)},$$

where $\Gamma(\cdot)$ denotes as usual the Gamma function.

The sharp affine L^2 Sobolev trace inequality on \mathbb{R}_+^n states that

Theorem 1. *Let $n \geq 3$. For any smooth function f with compact support on \mathbb{R}^n , we have*

$$(3) \quad \left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{2(n-1)}{n-2}} dx \right)^{\frac{n-2}{n-1}} \leq 2\mathcal{K}_n \mathcal{E}(f) \left(\int_{\mathbb{R}_+^n} \left| \frac{\partial f}{\partial t} \right|^2 dx dt \right)^{\frac{1}{2}},$$

where \mathcal{K}_n is the best constant for the trace inequality (1). Moreover, equality holds if, and only if,

$$f(t, x) = \pm ((\lambda t + \delta)^2 + |B(x - x_0)|^2)^{-\frac{n-2}{2}}$$

for some constants $\lambda, \delta > 0$, a point $x_0 \in \mathbb{R}^{n-1}$ and a matrix $B \in \text{GL}_{n-1}$, where GL_{n-1} denotes the set of invertible real $(n-1) \times (n-1)$ -matrices.

We remark that the inequality (3) is stronger than and implies the classical inequality (1). In fact, following the ideas used in [28], applying the Hölder inequality and Fubini's theorem to $\mathcal{E}(f)$, one easily deduces that $\mathcal{E}(f) \leq \|\tilde{\nabla} f\|_2$. This inequality clearly implies the claim.

The sharp affine L^2 Sobolev trace inequality is indeed a particular case of an affine L^p variant of Sobolev type trace inequalities.

Let $p > 1$. For smooth functions $f(t, x)$ with compact support on \mathbb{R}^n , denote

$$\begin{aligned}\mathcal{E}_p(f) &:= c_{n-1,p} \left(\int_{\mathbb{S}^{n-2}} \|\nabla_\xi f\|_{L^p(\mathbb{R}_+^n)}^{1-n} d\xi \right)^{\frac{1}{1-n}} \\ &= c_{n-1,p} \left(\int_{\mathbb{S}^{n-2}} \left(\int_{\mathbb{R}_+^n} |\nabla_\xi f(y)|^p dy \right)^{-\frac{n-1}{p}} d\xi \right)^{-\frac{1}{n-1}},\end{aligned}$$

where

$$c_{n,p} = (n\omega_n)^{\frac{1}{n}} \left(\frac{n\omega_n\omega_{p-1}}{2\omega_{n+p-2}} \right)^{\frac{1}{p}}.$$

Here we recall that for a real number $s > 0$, one defines

$$\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2} + 1)}.$$

The following theorem is a natural L^p extension of Theorem 1:

Theorem 2. *Let $1 < p < n$. For any smooth function f with compact support on \mathbb{R}^n , we have*

$$(4) \quad \left(\int_{\partial\mathbb{R}_+^n} |f(0, x)|^{\frac{p(n-1)}{n-p}} dx \right)^{\frac{n-p}{n-1}} \leq p\mathcal{A}_{n,p} \mathcal{E}_p(f)^{p-1} \left(\int_{\mathbb{R}_+^n} \left| \frac{\partial f}{\partial t} \right|^p dx dt \right)^{\frac{1}{p}},$$

where

$$\mathcal{A}_{n,p} = \pi^{-\frac{p-1}{2}} \left(\frac{(p-1)^{\frac{p-1}{p}}}{n-p} \right)^{p-1} \left(\frac{\Gamma(n)\Gamma(\frac{n+1}{2})}{(n-1)\Gamma(\frac{n-1}{p})\Gamma(\frac{n(p-1)+1}{p})} \right)^{\frac{p-1}{n-1}}$$

is sharp for this inequality. Moreover, equality holds if

$$f(t, x) = \pm ((\lambda t + \delta)^p + |B(x - x_0)|^p)^{-\frac{n-p}{p(p-1)}}$$

for some constants $\lambda, \delta > 0$, a point $x_0 \in \mathbb{R}^{n-1}$ and a matrix $B \in \text{GL}_{n-1}$.

Note that the inequality (4) is invariant under affine transformations of \mathbb{R}_+^n . In precise terms, denote by $\text{GL}_{n,+}$ the set of matrices of the form

$$(5) \quad \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & B & & \\ 0 & & & \end{pmatrix}$$

where $\lambda > 0$ and $B \in \text{GL}_{n-1}$. Then, the inequality (4) is $\text{GL}_{n,+}$ invariant. In particular, it does not depend on the Euclidean structure of \mathbb{R}_+^n .

The paper is organized as follows. In section 2 we fix some notations to be used in the paper and present the L_p Petty-Projection inequality. In section 3 we prove two central tools in our method (Lemmas 1 and 2). These two lemmas and the L_p Petty-Projection inequality are just used in the proof of Theorem 2 which is done in section 4. Finally, in section 4 we provide additional comments on the essence of our method and

on the characterization of extremal functions stated in Theorem 1. For convenience of the reader we include an appendix section devoted to more technical computations of a key constant (d_1) in the proof of Theorem 2.

2. PRELIMINARIES ON CONVEX BODIES

We recall that a convex body $K \subset \mathbb{R}^n$ is a convex compact subset of \mathbb{R}^n with non-empty interior.

For $K \subset \mathbb{R}^n$ as before, its support function h_K is defined as

$$h_K(y) = \max\{\langle y, z \rangle : z \in K\}.$$

The support function, which describes the (signed) distances of supporting hyperplanes to the origin, uniquely characterizes K . We also have the gauge $\|\cdot\|_K$ and radial $r_K(\cdot)$ functions of K defined respectively as

$$\begin{aligned} \|y\|_K &:= \inf\{\lambda > 0 : y \in \lambda K\}, \quad y \in \mathbb{R}^n \setminus \{0\}, \\ r_K(y) &:= \max\{\lambda > 0 : \lambda y \in K\}, \quad y \in \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

Clearly, $\|y\|_K = \frac{1}{r_K(y)}$. We also recall that $\|\cdot\|_K$ is actually a norm when the convex body K is symmetric (i.e. $K = -K$). On the other hand, any centrally symmetric convex body K is the unit ball for some norm in \mathbb{R}^n .

For $K \subset \mathbb{R}^n$ we define its polar body, denoted by K° , by

$$K^\circ := \{y \in \mathbb{R}^n : \langle y, z \rangle \leq 1 \quad \forall z \in K\}.$$

Evidently, $h_K = r_{K^\circ}$. It is also easy to see that $(\lambda K)^\circ = \frac{1}{\lambda} K^\circ$ for $\lambda > 0$. A simple computation using polar coordinates shows that

$$\text{vol}(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} r_K^n(y) dy = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \|y\|_K^{-n} dy.$$

For a given convex body $K \subset \mathbb{R}^n$ we find in the literature many associated bodies to it, in particular Lutwak and Zhang introduced [29] for a body K its L_p -centroid body denoted by $\Gamma_p K$. This body is defined by

$$h_{\Gamma_p K}^p(y) := \frac{1}{a_{n,p} \text{vol}(K)} \int_K |\langle y, z \rangle|^p dz \quad \text{for } y \in \mathbb{R}^n,$$

where

$$a_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

There are some other normalizations of the L_p -centroid body in the literature, the previous one is made so that $\Gamma_p \mathbb{B}^n = \mathbb{B}^n$ for the unit ball in \mathbb{R}^n centered at the origin.

Inequalities (usually affine invariant) that compare the volume of a convex body K and that of an associated body are common in the literature. For the specific case of K and $\Gamma_p K$, Lutwak, Yang and Zhang [27] (see also [8] for an alternative proof) came up with what it is known as the L_p Busemann-Petty centroid inequality, namely

$$(6) \quad \text{vol}(\Gamma_p K) \geq \text{vol}(K).$$

This inequality is sharp if and only if K is a 0-symmetric ellipsoid. For a comprehensive survey on L_p Brunn-Minkowski theory and other topics within Convex Geometry we refer to [31] and references therein.

3. FUNDAMENTAL LEMMAS

Let $C : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be an even convex function such that $C(x) > 0$ for $x \neq 0$. Assume also that C is q -homogeneous with $q > 1$, that is

$$C(\lambda y) = \lambda^q C(y), \quad \forall \lambda \geq 0, \quad y \in \mathbb{R}^n.$$

Denote by C^* its Legendre transform defined by

$$C^*(y) = \sup_{z \in \mathbb{R}^n} \{\langle y, z \rangle - C(z)\}.$$

One knows that C^* is also even, convex, p -homogeneous and positive for $x \neq 0$, where $\frac{1}{p} + \frac{1}{q} = 1$. Equivalently, these assumptions on C can be resumed by saying simply that C is a power q of an arbitrary norm on \mathbb{R}^n .

Let C be a function as before. Denote by

$$K_C = \{y \in \mathbb{R}^n : C(y) \leq 1\}.$$

It is easy to see that K_C is a centrally symmetric convex body with non-empty interior in \mathbb{R}^n . Moreover, it is defined by the norm $\|y\|_{K_C} = C(y)^{\frac{1}{q}}$.

The starting point as one of the essential ingredients in our work is the following theorem of [24].

Theorem 3. *Let $1 < p < n$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume C is even, convex, q -homogeneous and positive for $x \neq 0$. Then, for any smooth function f with compact support on \mathbb{R}^n , we have*

$$(7) \quad \left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{p(n-1)}{n-p}} dx \right)^{\frac{n-p}{n-1}} \leq \mathcal{K}_{n,C} \int_{\mathbb{R}_+^n} C^*(\nabla f) dx dt.$$

Moreover, equality holds in (7) if

$$(8) \quad f(t, x) = \gamma \left(\frac{1}{C(t + \delta, x - x_0)} \right)^{\frac{n-p}{p}}$$

for some real constants γ and δ with $\delta > 0$ and a point $x_0 \in \mathbb{R}^{n-1}$.

The proof of Theorem 2 is based on two lemmas and uses Theorem 3 in a crucial manner with the appropriate choice of C for each f , denoted by C_f , as stated in Definition 1 below.

Before introducing C_f , we state the first key tool.

Lemma 1. *Let p, q and C be as in Theorem 3. The optimal constant $\mathcal{K}_{n,C}$ is given by*

$$\mathcal{K}_{n,C} = p^p (n-p)^{-\frac{p}{q}} (q(n-1))^{-\frac{p}{q(n-1)}} \left(\int_{(K_C)_+} y_1^{qn-q-n} dy \right)^{-\frac{p}{q(n-1)}},$$

where $(K_C)_+ = K_C \cap \mathbb{R}_+^n$.

Proof. We calculate both left and right-hand sides of (7) with f given by (8), $\gamma = \delta = 1$ and $x_0 = 0$. Indeed, for

$$w_0(y) = C(y + e_1)^{-\frac{n-p}{q(p-1)}},$$

we have

$$\nabla w_0(y) = -\frac{n-p}{q(p-1)} C(y+e_1)^{-\frac{n-p}{q(p-1)}-1} \nabla C(y+e_1).$$

On the right-hand side, we have

$$\begin{aligned} C^*(\nabla w_0(y)) &= \left| -\frac{n-p}{q(p-1)} \right|^p C(y+e_1)^{-\frac{p(n-p)}{q(p-1)}-p} C^*(\nabla C(y+e_1)) \\ &= (q-1) \left(\frac{n-p}{q(p-1)} \right)^p C(y+e_1)^{1-n} \end{aligned}$$

and on the other hand

$$|w_0(y)|^{\frac{(n-1)p}{n-p}} = C(y+e_1)^{-\frac{n-p}{q(p-1)} \frac{(n-1)p}{n-p}} = C(y+e_1)^{1-n}.$$

Let us consider the parametrization of \mathbb{R}_+^n given by

$$\begin{cases} y = r\theta - e_1 \\ \theta \in \mathbb{S}_+^{n-1}, r \geq \theta_1^{-1} \end{cases}$$

with Jacobian r^{n-1} . Then,

$$\begin{aligned} \int_{\mathbb{R}_+^n} C^*(\nabla w_0(y)) dy &= (q-1) \left(\frac{n-p}{q(p-1)} \right)^p \int_{\mathbb{S}_+^{n-1}} \int_{\theta_1^{-1}}^{\infty} r^{n-1} C(r\theta)^{1-n} dr d\theta \\ &= (q-1) \left(\frac{n-p}{q(p-1)} \right)^p \int_{\mathbb{S}_+^{n-1}} \int_{\theta_1^{-1}}^{\infty} r^{-q(n-1)+n-1} C(\theta)^{1-n} dr d\theta \\ &= \frac{q-1}{q(n-1)-n} \left(\frac{n-p}{q(p-1)} \right)^p \int_{\mathbb{S}_+^{n-1}} \theta_1^{q(n-1)-n} C(\theta)^{1-n} d\theta. \end{aligned}$$

For the left-hand side of (7), we consider the parametrization of $\partial\mathbb{R}_+^n$,

$$\begin{cases} y = \theta_1^{-1}\theta - e_1 \\ \theta \in \mathbb{S}_+^{n-1} \end{cases}$$

with Jacobian θ_1^{-n} . Then,

$$\begin{aligned} \int_{\partial\mathbb{R}_+^n} |w_0(y)|^{\frac{(n-1)p}{n-p}} dy &= \int_{\mathbb{S}_+^{n-1}} \theta_1^{-n} C(\theta_1^{-1}\theta)^{1-n} d\theta \\ &= \int_{\mathbb{S}_+^{n-1}} \theta_1^{-n+q(n-1)} C(\theta)^{1-n} d\theta. \end{aligned}$$

Thus we have

$$\begin{aligned} \mathcal{K}_{n,C} &= \left(\frac{q-1}{q(n-1)-n} \left(\frac{n-p}{q(p-1)} \right)^p \right)^{-1} \left(\int_{\mathbb{S}_+^{n-1}} \theta_1^{-n+q(n-1)} C(\theta)^{1-n} d\theta \right)^{-\frac{p}{q(n-1)}} \\ &= p^p (n-p)^{-\frac{p}{q}} \left(\int_{\mathbb{S}_+^{n-1}} \theta_1^{-n+q(n-1)} C(\theta)^{1-n} d\theta \right)^{-\frac{p}{q(n-1)}} \end{aligned}$$

Let us note now that

$$\int_{(K_C)_+} y_1^{qn-q-n} dy = \int_{\mathbb{S}_+^{n-1}} \int_0^{r_K(\theta)} r^{n-1} (r\theta_1)^{qn-q-n} dr d\theta$$

$$\begin{aligned}
&= \int_{\mathbb{S}_+^{n-1}} \theta_1^{qn-q-n} \int_0^{r_K(\theta)} r^{qn-q-1} dr d\theta \\
&= \frac{1}{q(n-1)} \int_{\mathbb{S}_+^{n-1}} \theta_1^{qn-q-n} r_K(\theta)^{qn-q} d\theta = \frac{1}{q(n-1)} \int_{\mathbb{S}_+^{n-1}} \theta_1^{qn-q-n} C(\theta)^{1-n} d\theta.
\end{aligned}$$

So we obtain that

$$\begin{aligned}
\mathcal{K}_{n,C} &= p^p (n-p)^{-\frac{p}{q}} \left(q(n-1) \int_{(K_C)_+} y_1^{qn-q-n} dy \right)^{-\frac{p}{q(n-1)}} \\
&= p^p (n-p)^{-\frac{p}{q}} (q(n-1))^{-\frac{p}{q(n-1)}} \left(\int_{(K_C)_+} y_1^{qn-q-n} dy \right)^{-\frac{p}{q(n-1)}}.
\end{aligned}$$

□

Throughout the remainder of paper we think of \mathbb{R}^{n-1} as a subset $\{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$.

For each smooth function f with compact support on \mathbb{R}^n , consider

$$L_f = \{\xi \in \mathbb{R}^{n-1} : \|\nabla_\xi f\|_p \leq 1\},$$

which is a convex body in \mathbb{R}^{n-1} defined by the norm $\|\xi\|_{L_f} = \|\nabla_\xi f\|_p$.

For convenience, we set

$$(9) \quad Z_p(f) = \left(\int_{\mathbb{S}^{n-2}} \|\nabla_\xi f\|_p^{1-n} d\xi \right)^{\frac{1}{1-n}}$$

and notice we have the identities

$$(10) \quad (n-1) \text{vol}(L_f) = Z_p(f)^{1-n}$$

and

$$(11) \quad \mathcal{E}_p(f) = c_{n-1,p} Z_p(f).$$

We now are ready to introduce the definition of the function C_f .

Definition 1. Let f be a smooth function with compact support on \mathbb{R}^n . For $p > 1$ and $(t, x) \in \mathbb{R}_+^n$, we set

$$C_f^*(t, x) := \frac{\alpha_f}{p} |t|^p + \int_{\mathbb{S}^{n-2}} \|\nabla_\xi f\|_p^{1-n-p} |\langle x, \xi \rangle|^p d\xi,$$

where $\alpha_f = \frac{p}{p-1} \left\| \frac{\partial f}{\partial t} \right\|_p^{-p} Z_p(f)^{1-n}$. The function C_f is defined as the Legendre transform of C_f^* .

The specific choice of the constant α_f will be clarified at the last section.

Proposition 1. Let p and f be as in Definition 1. The function C_f^* is well defined, even, convex, p -homogeneous and $C_f^*(x) > 0$ for $x \neq 0$. Thus its Legendre transform C_f satisfies the hypotheses of Theorem 3.

Proof. The proof follows the same lines as in Proposition 1 in [20].

□

In order to simplify notation, for each f as before, we denote K_{C_f} by K_f , $(K_{C_f})_+$ by $(K_f)_+$, $\|\cdot\|_{K_{C_f}}$ by $\|\cdot\|_{K_f}$ and

$$D_f^*(x) := \int_{\mathbb{S}^{n-2}} \|\nabla_\xi f\|_p^{1-n-p} |\langle x, \xi \rangle|^p d\xi.$$

It is easy to see that

$$(12) \quad C_f(t, x) = \frac{\alpha_f^{1-q}}{q} |t|^q + D_f(x),$$

where D_f is the Legendre transform of D_f^* and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Let $K_{f,t} = \{x \in \mathbb{R}^{n-1} : (t, x) \in K_f\}$.

Using (12) we see that $K_{f,0} = \{x \in \mathbb{R}^{n-1} : D_f(x) \leq 1\}$ and

$$(13) \quad K_{f,t} = \left\{ x \in \mathbb{R}^{n-1} : D_f(x) \leq 1 - \frac{\alpha_f^{1-q}}{q} |t|^q \right\} = \left(1 - \frac{\alpha_f^{1-q}}{q} |t|^q \right)^{\frac{1}{q}} K_{f,0}.$$

The second central tool is stated as

Lemma 2. *Let p and f be as in Definition 1. The relation between $K_{f,0}$ and L_f is*

$$K_{f,0} = ((n+p-1)a_{n-1,p} \text{vol}(L_f))^{\frac{1}{p}} q^{\frac{1}{q}} p^{\frac{1}{p}} \Gamma_p L_f$$

and as a consequence,

$$(14) \quad \text{vol}(K_{f,0}) = \left(pq^{\frac{p}{q}} (n+p-1)a_{n-1,p} \right)^{\frac{n-1}{p}} \text{vol}(L_f)^{\frac{n-1}{p}} \text{vol}(\Gamma_p L_f),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Firstly, for $x \in \mathbb{R}^{n-1}$,

$$\begin{aligned} a_{n-1,p} \text{vol}(L_f) h_{\Gamma_p L_f}^p(x) &= \int_{L_f} |\langle x, w \rangle|^p dw \\ &= \int_{\mathbb{S}^{n-2}} \int_0^{r_{L_f}(\xi)} r^{n-2} |\langle x, r\xi \rangle|^p dr d\xi \\ &= \int_{\mathbb{S}^{n-2}} |\langle x, \xi \rangle|^p \int_0^{r_{L_f}(\xi)} r^{n-2+p} dr d\xi \\ &= \frac{1}{n+p-1} \int_{\mathbb{S}^{n-2}} |\langle x, \xi \rangle|^p r_{L_f}(\xi)^{n-1+p} d\xi \\ &= \frac{1}{n+p-1} \int_{\mathbb{S}^{n-2}} |\langle x, \xi \rangle|^p \|\nabla_\xi f\|_p^{1-n-p} d\xi \\ &= \frac{1}{n+p-1} D_f^*(x) \end{aligned}$$

Thus

$$(n+p-1)a_{n-1,p} \text{vol}(L_f) h_{\Gamma_p L_f}^p(x) = D_f^*(x),$$

so that

$$\begin{aligned}
(\Gamma_p L_f)^\circ &= \{x \in \mathbb{R}^{n-1} : h_{\Gamma_p L_f}^p(x) \leq 1\} \\
&= ((n+p-1)a_{n-1,p} \text{vol}(L_f))^{\frac{1}{p}} \{x \in \mathbb{R}^{n-1} : D_f^*(x) \leq 1\} \\
&= ((n+p-1)a_{n-1,p} \text{vol}(L_f))^{\frac{1}{p}} q^{\frac{1}{q}} p^{\frac{1}{p}} (K_{f,0})^\circ,
\end{aligned}$$

where in the last equality we used a general fact which we prove below for sake of completeness, once we were not able to find a proof in the literature. Finally, taking polar at both sides we obtain

$$\Gamma_p L_f = ((n+p-1)a_{n-1,p} \text{vol}(L_f))^{-\frac{1}{p}} q^{-\frac{1}{q}} p^{-\frac{1}{p}} K_{f,0},$$

and the proof of lemma follows. \square

Lemma 3. *Let $p > 1$ and $C : \mathbb{R}^m \rightarrow \mathbb{R}_+$, $m \geq 1$, be convex, q -homogeneous and positive for $x \neq 0$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$K_{C^*} = q^{\frac{1}{q}} p^{\frac{1}{p}} K_C^\circ.$$

Proof. Firstly, denote $\ell_q := \max_{t \geq 0} \{t^{\frac{1}{q}} - t\} = q^{-\frac{p}{q}} p^{-1}$. It suffices to prove that $K_C^\circ = \{x \in \mathbb{R}^m : C^*(x) \leq \ell_q\}$ and the lemma follows.

Take $x \in K_C^\circ$. Since C is q -homogeneous, we have for any $y \in \mathbb{R}^m \setminus \{0\}$ that $C(y)^{-\frac{1}{q}} y \in K_C$, thus

$$\begin{aligned}
\langle x, C(y)^{-\frac{1}{q}} y \rangle &\leq 1 \\
\langle x, y \rangle &\leq C(y)^{\frac{1}{q}} \\
\langle x, y \rangle - C(y) &\leq C(y)^{\frac{1}{q}} - C(y) \leq \ell_q,
\end{aligned}$$

so that $C^*(x) \leq \ell_q$.

Now take $x \in \mathbb{R}^m \setminus K_C^\circ$. By definition of K_C° , there exists $y \neq 0$ such that

$$\begin{aligned}
\langle x, y \rangle &> 1 \geq C(y) \\
\langle x, y \rangle &> C(y)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
\text{Hence, for any } t > 0, \quad \langle x, ty \rangle &> (t^q C(y))^{\frac{1}{q}} \\
\langle x, ty \rangle - C(ty) &> (t^q C(y))^{\frac{1}{q}} - t^q C(y).
\end{aligned}$$

Since $C(y) > 0$, we may choose $t > 0$ such that the right-hand side is maximized. Thus, we obtain $C^*(x) > \ell_q$. \square

4. PROOF OF THEOREM 2

We now prove Theorem 2 by using Lemmas 1 and 2 and the L_p Petty-Projection inequality.

From the definition of C_f , we see that

$$\begin{aligned}
(15) \quad \int_{\mathbb{R}_+^n} C_f^*(\nabla f(y)) dy &= \frac{\alpha_f}{p} \int_{\mathbb{R}_+^n} \left| \frac{\partial f}{\partial t} \right|^p dx dt + \int_{\mathbb{S}^{n-2}} \|\nabla_\xi f\|_p^{1-n} d\xi \\
&= \frac{q}{p} Z_p(f)^{1-n} + Z_p(f)^{1-n}
\end{aligned}$$

$$= qZ_p(f)^{1-n}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. From Lemma 1 and equation (13), we have

$$\begin{aligned}
 (16) \quad \mathcal{K}_{n,C_f} &= p^p(n-p)^{-\frac{p}{q}}(q(n-1))^{-\frac{p}{q(n-1)}} \left(\int_{(K_f)_+} y_1^{qn-q-n} dy \right)^{-\frac{p}{q(n-1)}} \\
 &= p^p(n-p)^{-\frac{p}{q}}(q(n-1))^{-\frac{p}{q(n-1)}} \left(\int_0^{q^{\frac{1}{q}}\alpha_f^{\frac{1}{p}}} t^{qn-q-n} \left(1 - \frac{\alpha_f^{1-q}}{q} t^q \right)^{\frac{n-1}{q}} \text{vol}(K_{f,0}) dt \right)^{-\frac{p}{q(n-1)}} \\
 &= p^p(n-p)^{-\frac{p}{q}}(q(n-1))^{-\frac{p}{q(n-1)}} \text{vol}(K_{f,0})^{-\frac{p}{q(n-1)}} q^{-\frac{p}{q}} \alpha_f^{-1} \left(\int_0^1 t^{qn-q-n} (1-t^q)^{\frac{n-1}{q}} dt \right)^{-\frac{p}{q(n-1)}} \\
 &= p^p(n-p)^{-\frac{p}{q}}(q(n-1))^{-\frac{p}{q(n-1)}} \left(\frac{1}{q} \frac{\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} q^{-\frac{1}{q}} \text{vol}(K_{f,0})^{-\frac{p}{q(n-1)}} \alpha_f^{-\frac{1}{p}},
 \end{aligned}$$

The latter using the fact that

$$\int_0^1 t^{qn-q-n} (1-t^q)^{\frac{n-1}{q}} dt = \frac{1}{q} \frac{\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{\Gamma(n)}.$$

Putting together (15) and (16), we obtain

$$\begin{aligned}
 \mathcal{K}_{n,C_f} \int_{\mathbb{R}_+^n} C_f^*(\nabla f(y)) dy &= \left(pq^{-\frac{1}{pq}}(n-p)^{-\frac{1}{q}}(q(n-1))^{-\frac{1}{q(n-1)}} \left(\frac{\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{q\Gamma(n)} \right)^{-\frac{1}{q(n-1)}} \right)^p \\
 &\quad \times \text{vol}(K_{f,0})^{-\frac{p}{q(n-1)}} \alpha_f^{-\frac{1}{p}} qZ_p(f)^{1-n}.
 \end{aligned}$$

By Lemma 2, the L_p Busemann-Petty centroid inequality (6), (9), (10) and (11),

$$\mathcal{K}_{n,C_f} \int_{\mathbb{R}_+^n} C_f^*(\nabla f(y)) dy \leq d_1 \mathcal{E}_p(f)^{p-1} \left(\int_{\mathbb{R}_+^n} \left| \frac{\partial f}{\partial t} \right|^p dx dt \right)^{\frac{1}{p}},$$

where a careful computation of d_1 (that can be found in the appendix) gives the constant in the statement of the Theorem.

Finally, we obtain

$$\begin{aligned}
 (17) \quad \left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{p(n-1)}{n-p}} dx \right)^{\frac{n-p}{n-1}} &\leq \mathcal{K}_{n,C_f} \int_{\mathbb{R}_+^n} C_f^*(\nabla f) dx dt \\
 &\leq d_1 \mathcal{E}_p(f)^{p-1} \left(\int_{\mathbb{R}_+^n} \left| \frac{\partial f}{\partial t} \right|^p dx dt \right)^{\frac{1}{p}}.
 \end{aligned}$$

□

5. FURTHER COMMENTS

As mentioned in introduction, the same ideas used in [28] to estimate $\mathcal{E}(f)$, involving Hölder's inequality and Fubini's theorem, produce $\mathcal{E}_p(f) \leq \|\tilde{\nabla} f\|_p$. Thus, using the Young inequality, we get

$$(18) \quad \left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{p(n-1)}{n-p}} dx \right)^{\frac{n-p}{n-1}} \leq d_2 \left(\mathcal{E}_p(f)^p + \left\| \frac{\partial f}{\partial t} \right\|_p^p \right) \\ \leq d_2 \left(\int_{\mathbb{R}_+^n} |\tilde{\nabla} f|^p + \left| \frac{\partial f}{\partial t} \right|^p dx dt \right)$$

with $d_2 = (p^{\frac{1}{p}} q^{\frac{1}{q}})^{-\frac{1}{p}} d_1$.

Observe that the inequality $\mathcal{E}_p(f) \leq \|\tilde{\nabla} f\|_p$ becomes an equality for functions which are radial with respect to the variable x . On the other hand, taking $C^*(t, x) = |t|^p + |x|^p$ with Euclidean norm in x , Theorem 3 provides the sharp inequality

$$\left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{p(n-1)}{n-p}} dx \right)^{\frac{n-p}{n-1}} \leq \mathcal{K}_{n,C} \int_{\mathbb{R}_+^n} C^*(\nabla f) dx dt \\ \leq \mathcal{K}_{n,C} \left(\int_{\mathbb{R}_+^n} \left| \frac{\partial f}{\partial t} \right|^p + |\tilde{\nabla} f|^p dx dt \right)$$

and a careful calculation of the constant above using the formula of Lemma 1 gives exactly d_2 . This is the reason for the choice of the constants in Definition 1. Since the extremal functions (8) corresponding to the previous inequality are radial with respect to x , we conclude that the inequalities (18) and thus (17) are sharp.

Now if f is an extremal function of equation (4) then the equality on $\text{vol}(K_{f,0}) = \text{vol}(\Gamma_p K_{f,0})$ implies that $K_{f,0}$ is an ellipsoid. Thus after a $\text{GL}_{n,+}$ change of coordinates, we may assume $K_{f,0}$ is a ball and C_f^* is the function C^* defined in the above paragraph.

For the special case $p = 2$, the extremal functions were characterized by Escobar in [11]. Then the extremal functions of equation (3) are exactly the ones described in Theorem 1.

Since the characterization of the extremal functions in the general case $1 < p < n$ was left open by Nazaret in [24] we cannot conclude that f is of the form (8). But this is the only step left to characterize the extremal functions of (4).

In the case the aforementioned characterization is done, there is an other important conclusion and that is that (4) is not stronger than the classical sharp trace inequality

$$\left(\int_{\partial \mathbb{R}_+^n} |f(0, x)|^{\frac{p(n-1)}{n-p}} dx \right)^{\frac{n-p}{n-1}} \leq \mathcal{K}_{n,p} \int_{\mathbb{R}_+^n} |\nabla f(t, x)|^p dx dt$$

simply because they have different families of extremal functions.

REFERENCES

- [1] Aubin, T. - *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geom. 11 (1976) 573-598.
- [2] Beckner, W. - *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. Math. 138 (1993) 213-242.

- [3] Brendle, S., Chen, S. S. - *An existence theorem for the Yamabe problem on manifolds with boundary*, J. Eur. Math. Soc. (JEMS) 16 (2014) 991-1016.
- [4] Cabré, X., Tan, J. - *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Advances in Mathematics 224 (2010), 2052-2093.
- [5] Caffarelli, L., Roquejoffre, J.M., Sire, Y. - *Variational problems in free boundaries for the fractional Laplacian*, J. Eur. Math. Soc. 12 (2010), 1151-1179.
- [6] Caffarelli, L., Silvestre, L. - *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), 1245-1260.
- [7] Capella, A., Dávila, J., Dupaigne, L., Sire, Y. - *Regularity of Radial Extremal Solutions for Some Non-Local Semilinear Equations*, Comm. Partial Differential Equations 36 (2011) 1353-1384.
- [8] Campi, S., Gronchi, P. - *The L_p -Busemann-Petty centroid inequality*. Adv. Math. 167 (2002), pp. 128-141.
- [9] Cianchi, A., Ferone V., Nitsch, C., Trombetti, C. - *Balls minimize trace constants in BV* . Journal für die reine und angewandte Mathematik (Crelles Journal). (2014) doi:10.1515/crelle-2014-0098
- [10] Cianchi, A. - *A sharp trace inequality for functions of bounded variation in the ball*. Proceedings of the Royal Society of Edinburgh. Section A. Mathematics, vol. 142, pp. (2012) 1179-1191
- [11] Escobar, J.F. - *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J., 37 (1988), pp. 687-698.
- [12] Escobar, J.F. - *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. 136 (1992) 1-50.
- [13] Escobar, J.F. - *The Yamabe problem on manifolds with boundary*, J. Differential Geom. 35 (1992) 21-84.
- [14] Federer, H., Fleming, W.H. - *Normal and integral currents*, Ann. of Math. 72 (1960) 458-520.
- [15] Fernández Bonder, J., Saintier, N. - *Estimates for the Sobolev trace constant with critical exponent and applications*. Ann. Mat. Pura Appl. (4), 187 (2008), no. 4, 683-704.
- [16] Fleming, W.H., Rishel, R. - *An integral formula for total gradient variation*, Arch. Math. 11 (1960) 218-222.
- [17] C. Haberl, F.E. Schuster - *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. 257 (2009), 641-658.
- [18] C. Haberl, F.E. Schuster - *General L_p affine isoperimetric inequalities*, J. Differential Geom. 83 (2009), 1-26.
- [19] C. Haberl, F.E. Schuster, J. Xiao - *An asymmetric affine Polya-Szegoe principle*, Math. Annalen 352 (2012) 517-542.
- [20] Haddad, J., Jiménez, C.H., Montenegro, M. - *Sharp affine Sobolev type inequalities via the L_p Busemann-Petty centroid inequality*, arXiv:1505.07763v1 28 May 2015.
- [21] Han, Z.-C., Li, Y.Y. - *The Yamabe problem on manifolds with boundary: existence and compactness results*, Duke Math. J. 99 (1999) 489-542.
- [22] Marques, F.C. - *Existence results for the Yamabe problem on manifolds with boundary*, Indiana Univ. Math. J. 54 (2005) 1599-1620.
- [23] Maz'ja, V. G. - *Classes of domains and imbedding theorems for function spaces*, Soviet Math. Dokl. 1 (1960) 882-885.
- [24] Nazaret, B. - *Best constant in Sobolev trace inequalities on the half-space*, Nonlinear Analysis TMA 65 (2006) 1977-1985.
- [25] Lions, P.L. - *The concentration-compactness principle in the calculus of variations. The locally compact case II*, Rev. Mat. Iberoamericana 1 (2) (1985) 45-121.
- [26] Lutwak, E. - *On some affine isoperimetric inequalities*, J. Differential Geom. 23 (1986) 1-13.
- [27] Lutwak, E., Yang, D., Zhang, G. - *L_p affine isoperimetric inequalities*. J. Differential Geom. 56 (2000) 111-132.
- [28] Lutwak, E., Yang, D., Zhang, G. - *Sharp affine L_p Sobolev Inequalities*. J. Differential Geom. 62 (2002) 17-38.
- [29] Lutwak, E., Zhang, G. - *Blaschke-Santaló inequalities*. J. Differential Geom. 47 (1997) 1-16.
- [30] Petty, C.M. - *Centroid surfaces*, Pacific J. Math. 11 1961 1535-1547. 52.30 (53.90)
- [31] Schneider, R. - *Convex Bodies: The Brunn-Minkowski Theory*. 2nd ed. Cambridge: Cambridge University Press, 2013.
- [32] Silvestre, L. - *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Commun. Pure Appl. Math. 60 (2007), 67-112.
- [33] Talenti, G. - *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (iv) 110 (1976) 353-372.
- [34] Zhang, G. - *The affine Sobolev inequality*, J. Differential geometry **53** (1999), pp. 183-202.

6. APPENDIX

In this appendix, for sake of completeness, we compute the constant d_1 that appears in the proof of Theorem 2.

Firstly,

$$\begin{aligned}
\mathcal{L}_{C_f}^p \int_{\mathbb{R}_+^n} C_f^*(\nabla f(y)) dy &= \left(pq^{-\frac{1}{p}} (n-p)^{-\frac{1}{q}} (q(n-1))^{-\frac{1}{q(n-1)}} \left(\frac{\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{q\Gamma(n)} \right)^{-\frac{1}{q(n-1)}} \right)^p \\
&\times \text{vol}(K_{f,0})^{-\frac{p}{q(n-1)}} \alpha_f^{-\frac{1}{p}} q Z_p(f)^{1-n} \\
&= p^p q^{-\frac{1}{q}} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
&\text{(using (14))} \times \left[\left(pq^{\frac{p}{q}} (n+p-1) a_{n-1,p} \right)^{\frac{n-1}{p}} \text{vol}(L_f)^{\frac{n-1}{p}} \text{vol}(\Gamma_p L_f) \right]^{-\frac{p}{q(n-1)}} \\
&\times \alpha_f^{-\frac{1}{p}} q Z_p(f)^{1-n} \\
&= p^p q^{-\frac{1}{q}} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
&\times \left(pq^{\frac{p}{q}} (n+p-1) a_{n-1,p} \right)^{-\frac{1}{q}} \text{vol}(L_f)^{-\frac{1}{q}} \text{vol}(\Gamma_p L_f)^{-\frac{p}{q(n-1)}} \\
&\times \alpha_f^{-\frac{1}{p}} q Z_p(f)^{1-n} \\
&\text{(Using B-P ineq for } L_f) \leq p^{p-\frac{1}{q}} q^{-\frac{1}{q}-\frac{p}{q^2}+1} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
&\times ((n+p-1) a_{n-1,p})^{-\frac{1}{q}} \text{vol}(L_f)^{-\frac{p+n-1}{q(n-1)}} \\
&\times \alpha_f^{-\frac{1}{p}} Z_p(f)^{1-n} \\
&= p^{p-\frac{1}{q}} q^{-\frac{1}{q}-\frac{p}{q^2}+1} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
&\text{(Using (10))} \times ((n+p-1) a_{n-1,p})^{-\frac{1}{q}} \left(\frac{1}{n-1} c_{n-1,p}^{n-1} \right)^{-\frac{p+n-1}{q(n-1)}} \\
&\times \mathcal{E}_p(f)^{\frac{n+p-1}{q}} \alpha_f^{-\frac{1}{p}} Z_p(f)^{1-n} \\
&= p^{p-\frac{1}{q}} q^{-\frac{1}{q}-\frac{p}{q^2}+1} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right) \Gamma\left(\frac{n-1}{q} + 1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}}
\end{aligned}$$

$$\begin{aligned}
 & \times ((n+p-1)a_{n-1,p})^{\frac{-1}{q}} \left(\frac{1}{n-1} c_{n-1,p}^{n-1} \right)^{-\frac{p+n-1}{q(n-1)}} \\
 (\text{By definition of } \alpha_f) & \times \mathcal{E}_p(f)^{\frac{n+p-1}{q}} \left(q \left\| \frac{\partial f}{\partial t} \right\|_p^{-p} Z_p(f)^{1-n} \right)^{-\frac{1}{p}} Z_p(f)^{1-n} \\
 & = p^{p-\frac{1}{q}} q^{-\frac{p}{q^2}} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n-1}{q}+1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
 & \times ((n+p-1)a_{n-1,p})^{\frac{-1}{q}} \left(\frac{1}{n-1} c_{n-1,p}^{n-1} \right)^{-\frac{p+n-1}{q(n-1)}} \\
 & \times \mathcal{E}_p(f)^{\frac{n+p-1}{q}} \left\| \frac{\partial f}{\partial t} \right\|_p Z_p(f)^{1-n+\frac{n-1}{p}} \\
 & = p^{p-\frac{1}{q}} q^{-\frac{p}{q^2}} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n-1}{q}+1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
 & \times ((n+p-1)a_{n-1,p})^{\frac{-1}{q}} \left(\frac{1}{n-1} c_{n-1,p}^{n-1} \right)^{-\frac{p+n-1}{q(n-1)}} \\
 (\text{Using (11)}) & \times \mathcal{E}_p(f)^{\frac{n+p-1}{q}} \left\| \frac{\partial f}{\partial t} \right\|_p c_{n-1,p}^{(n-1)-\frac{n-1}{p}} \mathcal{E}_p(f)^{1-n+\frac{n-1}{p}} \\
 & = p^{p-\frac{1}{q}} q^{-\frac{p}{q^2}} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n-1}{q}+1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
 & \times (n+p-1)^{\frac{-1}{q}} (a_{n-1,p})^{\frac{-1}{q}} \left(\frac{1}{n-1} \right)^{-\frac{p+n-1}{q(n-1)}} \\
 & \times \left\| \frac{\partial f}{\partial t} \right\|_p c_{n-1,p}^{(n-1)-\frac{n-1}{p}-\frac{p+n-1}{q}} \mathcal{E}_p(f)^{1-n+\frac{n-1}{p}+\frac{n+p-1}{q}} \\
 & = p^{p-\frac{1}{q}} q^{-\frac{p}{q^2}} (n-p)^{-p/q} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n-1}{q}+1\right)}{\Gamma(n)} \right)^{-\frac{p}{q(n-1)}} \\
 & \times (n+p-1)^{\frac{-1}{q}} (a_{n-1,p})^{\frac{-1}{q}} \left(\frac{1}{n-1} \right)^{-\frac{p+n-1}{q(n-1)}} c_{n-1,p}^{\frac{-p}{q}} \\
 & \times \left\| \frac{\partial f}{\partial t} \right\|_p \mathcal{E}_p(f)^{\frac{p}{q}}.
 \end{aligned}$$

Let us now denote

$$\mathcal{A}_{n,p} = \frac{1}{p} \times p^{p-\frac{1}{q}} q^{-\frac{p}{q^2}} (n-p)^{-\frac{p}{q}} (n+p-1)^{-\frac{1}{q}} \left(\frac{1}{n-1} \right)^{-\frac{n+p-1}{(n-1)q}}$$

$$\times \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n-1}{q}+1\right)}{\Gamma(n)} \right)^{-\frac{p}{(n-1)q}} \left(a_{n-1,p} c_{n-1,p}^p \right)^{-\frac{1}{q}}.$$

But,

$$\begin{aligned} a_{n-1,p} c_{n-1,p}^p &= \frac{\left(2^{-\frac{1}{p}} ((n-1)\omega_{n-1})^{\frac{1}{n-1}} \left(\frac{(n-1)\omega_{n-1}\omega_{p-1}}{\omega_{n+p-3}} \right)^{\frac{1}{p}} \right)^p \omega_{n+p-1}}{\omega_2 \omega_{n-1} \omega_{p-1}} \\ &= \frac{(n-1) ((n-1)\omega_{n-1})^{\frac{p}{n-1}} \omega_{n+p-1}}{2\omega_2 \omega_{n+p-3}} \\ &= \frac{\pi(n-1) ((n-1)\omega_{n-1})^{\frac{p}{n-1}}}{\omega_2(n+p-1)} \\ &= \frac{\pi(n-1)^{\frac{n+p-1}{n-1}} \omega_{n-1}^{\frac{p}{n-1}}}{\omega_2(n+p-1)}. \end{aligned}$$

Then we obtain that

$$\begin{aligned} \mathcal{A}_{n,p} &= p^{p-1-\frac{1}{q}} q^{-\frac{p}{q^2}} \left(\frac{1}{n-1} \right)^{-\frac{n+p-1}{(n-1)q}} (n-p)^{-\frac{p}{q}} (n+p-1)^{-\frac{1}{q}} \\ &\times \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n-1}{q}+1\right)}{\Gamma(n)} \right)^{-\frac{p}{(n-1)q}} \left(\frac{\pi^{p/2} (n-1)^{\frac{p}{n-1}+1} \Gamma\left(\frac{n+1}{2}\right)^{\frac{p}{1-n}}}{n+p-1} \right)^{-\frac{1}{q}} \\ &= p^{p-1-\frac{1}{q}} \pi^{-\frac{p}{2q}} q^{-\frac{p}{q^2}} (n-p)^{-\frac{p}{q}} \left(\frac{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n-1}{q}+1\right)}{\Gamma(n)\Gamma\left(\frac{n+1}{2}\right)} \right)^{-\frac{p}{(n-1)q}} \\ &= p^{\frac{1}{p}-1} \pi^{\frac{1}{2}-\frac{p}{2}} q^{\frac{1}{q}} \left(\frac{n-p}{p-1} \right)^{1-p} \left(\frac{\Gamma(n)\Gamma\left(\frac{n+1}{2}\right)}{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n}{q}+\frac{1}{p}\right)} \right)^{\frac{p-1}{n-1}} \\ &= \pi^{-\frac{p-1}{2}} \left(\frac{(p-1)^{\frac{p-1}{p}}}{n-p} \right)^{p-1} \left(\frac{\Gamma(n)\Gamma\left(\frac{n+1}{2}\right)}{(n-1)\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n(p-1)+1}{p}\right)} \right)^{\frac{p-1}{n-1}}, \end{aligned}$$

where $d_1 = p\mathcal{A}_{n,p}$ is the optimal constant that appears in our main theorem.

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